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ARITHMETIC CALCULUS OF FOURIER TRANSFORMS BY IGUSA LOCAL ZETA FUNCTIONS

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ABSTRACT. We show the possibility of explicit calculation of the Fourier transforms of complex powers of relative invariants of some prehomogeneous vector spaces over $\mathbb{R}$ by using the explicit form of p-adic Igusa local zeta functions.

Let (G, ρ, V) be a regular prehomogeneous vector space defined over $\mathbb{R}$, and $f_1, \dots, f_r$ the basic $\mathbb{R}$-relative invariants of (G, ρ, V) (cf. [F. Sato 2, p. 444]). For explicit calculation of Fourier transforms of the complex power

$$|f(x)|^s = \prod_{i=1}^{r} |f_i(x)|^{s_i}$$

with $s = (s_1, \dots, s_r) \in \mathbb{C}^r$, two general methods are known. When (G, ρ, V) has finitely many G-orbits, one can use a microlocal calculus established by M. Kashiwara according to M. Sato's idea (see p. 404 in [M], [K–K–M]). When $r = 1$ and the relative invariant is linear for each variable, one can use Igusa's method (see p. 8 in [Igusa 2]).

In this paper, we shall show the possibility of calculation for some cases by using the explicit form of the Igusa's local zeta function, based on the idea of Iwasawa-Tate theory [Iwasawa], [Tate].

As an example, we shall calculate the Fourier transform of $|f(x)|^s$ for $(GL_1^4 \times SL_{2m+1},\ \Lambda_2 \oplus \Lambda_1 \oplus \Lambda_1^* \oplus \Lambda_1^*,\ \mathrm{Alt}_{2m+1} \oplus \mathrm{Aff}^{2m+1} \oplus \mathrm{Aff}^{2m+1} \oplus \mathrm{Aff}^{2m+1})$ which has infinitely many orbits and $r = 4$ so that it has not been calculated by other methods.

The Igusa local zeta function of this prehomogeneous vector space has been explicitly calculated by [Hosokawa]. Our method is applicable for a prehomogeneous vector space with $Z_a = \tau Z_m$ for some constant $\tau > 0$. For example, irreducible (resp. simple, 2-simple of type I) regular prehomogeneous vector spaces with finitely many adelic open orbits satisfy $Z_a = \tau Z_m$ (see [Igusa 4] and [K–K]).

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1. Preliminaries

Let $\tilde{G}$ be a connected linear algebraic group and $\rho : \tilde{G} \to GL(V)$ a rational representation of $\tilde{G}$ on a finite-dimensional vector space V. Put $G = \rho(\tilde{G}) \subset GL(V)$. When V has a Zariski-dense G-orbit Y, we call a triplet $(\tilde{G}, \rho, V)$ (or a pair (G, V)) a prehomogeneous vector space (abbrev. P.V.). A point of Y is called a generic point and the isotropy subgroup at a generic point is called a generic isotropy subgroup which is unique up to isomorphisms. The complement S of Y is a Zariski-closed set which is called the singular set of (G, V). An irreducible component S_i of codimension one is the zeros of some irreducible polynomial $f_i(x)$ $(i = 1, \ldots, r)$. Then these polynomials are algebraically independent relative invariants, i.e., $f_i(\rho(g)x) = \chi_i(g)f_i(x)$ for $g \in \tilde{G}$ and $x \in V$ with some rational characters χ_i of $\tilde{G}$. Moreover any relatively invariant rational function $f(x)$ is of the form $f(x) = cf_1(x)^{m_1} \cdots f_r(x)^{m_r}$ with $(m_1, \ldots, m_r) \in \mathbb{Z}^r$ and some constant c (see p. 60 in [S–K]). We call $f_1, \ldots, f_r$ the basic relative invariants of $(\tilde{G}, \rho, V)$. Now a P.V. is called regular if the Hessian $H_f(x) = \det(\frac{\partial^2 f}{\partial x_i \partial x_j}(x))$ is not identically zero for some relative invariant $f(x)$. In this case, we have $\det \rho(g)^2 = \chi_1(g)^{2\kappa_1} \cdots \chi_r(g)^{2\kappa_r}$ for some $\kappa = (\kappa_1, \ldots, \kappa_r) \in (\frac{1}{2}\mathbb{Z})^r$ (see p. 61 in [S–K]). When $r = 1$, we have $\kappa = \frac{n}{d}$ with $d = \deg f$ and $n = \dim V$.

When G is reductive, it is regular if and only if a generic isotropy subgroup is reductive, and it is so if and only if the singular set S is a hypersurface (see p. 73 in [S–K]). In this case, without essential loss of generality, we may assume that a generic isotropy subgroup is semisimple (cf. §3 and §4 in [K–K]). Let k be an algebraic number field. From now on, we assume that (G, V) is a reductive P.V. defined over k with a connected semisimple generic isotropy subgroup, and all coefficients of $f_i(x)$ are in k. We denote by G_A, V_A, etc., the adelization of G, V, etc., with respect to k. Let $\Omega(k_A^\times/k^\times)$ be the space of quasicharacters of the idele class group $k_A^\times/k^\times$ of k and $\mathfrak{S}(V_A)$ the Schwarz-Bruhat space of V_A. For $\omega = (\omega_1, \ldots, \omega_r) \in \Omega(k_A^\times/k^\times)^r$, we write $\omega(\chi(g)) = \omega_1(\chi_1(g)) \cdots \omega_r(\chi_r(g))$ and $\omega(f(x)) = \omega_1(f_1(x)) \cdots \omega_r(f_r(x))$ ($g \in G_A$, $x \in Y_A$) for simplicity. Now we define the two adelic zeta-functions $Z_a(\omega, \Phi)$ and $Z_m(\omega, \Phi)$ of (G, V).

$$(1.1) \qquad Z_a(\omega, \Phi) = \int_{G_A/G_k} \omega(\chi(g)) \sum_{\xi \in Y_k} \Phi(g\xi) d_{G_A}(g),$$

$$(1.2) \qquad Z_m(\omega, \Phi) = \int_{Y_A} \omega(f(x)) \Phi(x) d_{Y_A}(x) \quad (\Phi \in \mathfrak{S}(V_A)).$$

Here d_{G_A} is a right-invariant measure on G_A while d_{Y_A} is a G_A-invariant measure on Y_A. Since a generic isotropy subgroup is connected semisimple, we may take the same convergence factor for d_{G_A} and d_{Y_A}. Note that, for the simplest P.V.(GL_1, Aff^1), we have

$$Z_m = \int_{k_A^\times} \omega(x)\Phi(x)d^\times x = \int_{k_A^\times/k^\times} \omega(x) \sum_{\xi\in k^\times} \Phi(x\xi)d^\times x = Z_a$$

which appears in the original Iwasawa-Tate theory. For $s = (s_1, \ldots, s_r) \in \mathbb{C}^r$ and $x = (x_1, \ldots, x_r) \in {k_A^\times}^r$, we put $\omega_s(x) = |x_1|_A^{s_1} \cdots |x_r|_A^{s_r}$. For any $\omega \in \Omega(k_A^\times/k^\times)^r$, we have $|\omega(x)| = \omega_\sigma(x)$ for some $\sigma = (\sigma_1, \ldots, \sigma_r) \in \mathbb{R}^r$. In this case, we write $\sigma(\omega) = (\sigma_1, \ldots, \sigma_r) \in \mathbb{R}^r$. It is known that $Z_m(\omega, \Phi)$ is absolutely convergent when $\sigma(\omega) = (\sigma_1, \ldots, \sigma_r) > \kappa = (\kappa_1, \ldots, \kappa_r)$, i.e., $\sigma_i > \kappa_i$ for $i = 1, \ldots, r$ (see [Ono], p. 90 in [F. Sato 1]). One can see that $Z_m(\omega, \Phi)$ has an Euler product $Z_m(\omega, \Phi) = \prod_{v\in\Sigma} Z_v(\omega_v, \Phi_v)$ for $\Phi = \bigotimes_{v\in\Sigma}\Phi_v$ where Σ denotes the set of places of k. We can express the local factor $Z_v(\omega_v, \Phi_v)$ by the Igusa local zeta function for almost all finite places v. We define $Z_v(s)$ to be the Igusa local zeta function $\int_{O_v^n} |f_1(x)|^{s_1} \cdots |f_r(x)|^{s_r} dx$ with $\int_{O_v^n} dx = 1$. It is a rational function of $t_i = q_v^{-s_i}$ $(i = 1, \ldots, r)$.

Theorem 1.1 ([Igusa 4], [K–K]). *Let (G, ρ, V) be an irreducible (resp. simple, 2-simple of type* I) *P.V. defined and split over an algebraic number field k satisfying $|G_A\backslash Y_A| < +\infty$. Then we have $Z_a(\omega, \Phi) = \tau Z_m(\omega, \Phi)$ for some constant $\tau > 0$.*

From now on, we assume that $Z_a = \tau Z_m$ for some $\tau > 0$ so that $Z_a(\omega, \Phi)$ converges absolutely when $\sigma(\omega) > \kappa$.

2. Functional equation

We have assumed that $(\tilde{G}, \rho, V)$ or (G, V) is a reductive regular P.V. defined and split over an algebraic number field with a connected semisimple generic isotropy subgroup and $Z_a(\omega, \Phi) = \tau Z_m(\omega, \Phi)$ for some $\tau > 0$. We may assume that $f_i(x) \in O_k[x]$ $(i = 1, \ldots, r)$ where O_k is the maximal order of k. Let ρ^* be the contragredient representation of ρ on the dual vector space V^* of V. Then $(\tilde{G}, \rho^*, V^*)$ is also a reductive regular P.V. defined over k with the singular set S^*, and $Y^* = V^* - S^*$. Since $\tilde{G}$ is reductive, we have the basic relative invariants $f_1^*, \ldots, f_r^*$ satisfying

$$f_i^*(\rho^*(g)y) = \chi_i(g)^{-1} f_i^*(y) \qquad (i = 1, \ldots, r)$$

for $g \in \tilde{G}$ and $y \in V^*$. By taking a basis, compatible with the k-structure of (G, ρ, V), we may identify V and V^* with Aff^n so that $\rho^*(\tilde{g}) = {}^t\rho(\tilde{g})^{-1}$ for $\tilde{g} \in \tilde{G}$. We write $g^* \cdot y = {}^tg^{-1}y$ for $g = \rho(\tilde{g}) \in G = \rho(\tilde{G})$, and $y \in V^*$. We define $Z_a^*(\omega, \Psi)$ and $Z_m^*(\omega, \Psi)$ as follows:

$$(2.1) \qquad Z_a^*(\omega, \Psi) = \int_{G_A/G_k} \omega(\chi^{-1}(g)) \sum_{\eta\in Y_k^*} \Psi(g^*\eta)d_{G_A}(g),$$

$$(2.2)\qquad Z_m^*(\omega, \Psi) = \int_{Y_A^*} \omega(f^*(y))\Psi(y)d_{Y_A^*}(y) \qquad (\Psi \in \mathfrak{S}(V_A^*)).$$

For any place $v \in \Sigma$ of k, let k_v be the local field corresponding to v. For $\Phi_v \in \mathfrak{S}(V_v)$ with $V_v = k_v^n$, let $\hat{\Phi}_v$ be its Fourier transform with respect to the self-dual measure so that $\hat{\hat{\Phi}}_v(x) = \Phi_v(-x)$ holds. For a finite place v, the self-dual measure $d_1 x$ satisfies $\int_{O_v^n} d_1 x = N(d_v)^{-\frac{n}{2}}$ where d_v is the different of k_v, and hence note that it is not the measure dx satisfying $\int_{O_v^n} dx = 1$ which appears in the definition of the Igusa local zeta function. For $\omega_v = (\omega_v^1, \dots, \omega_v^r)$ where ω_v^i is a quasicharacter of $k_v^\times$, we define the local zeta function Z_v by

$$Z_v(\omega_v, \Phi_v) \overset{\text{def}}{=} \int_{Y_v} \omega_v(f(x))\Phi_v(x)d_{Y_v}(x).$$

Similarly we define Z_v^* by

$$Z_v^*(\omega_v, \Psi_v) \overset{\text{def}}{=} \int_{Y_v^*} \omega_v(f^*(y))\Psi_v(y)d_{Y_v^*}(y).$$

Lemma 2.1. *For any infinite place $v \in \Sigma_\infty$, there exists $\Phi_v \in \mathfrak{S}(V_v)$ satisfying $\Phi_v \neq 0$, $\Phi_v|_{S_v} = 0$, and $\hat{\Phi}_v|_{S_v^*} = 0$.*

Proof. Put $F^* = f_1^* \cdots f_r^*$ so that S^* is the zeros of F^*. Take a nonzero $\Phi_o \in C_0^\infty(Y_v)$ and put $\Phi_v = F^*(\mathrm{grad}_x)\Phi_0$. Since $\hat{\Phi}_v(y) = \pm F^*(y)\hat{\Phi}_0(y)$, this Φ_v satisfies our conditions. This is a well-known argument. □

We denote by A_f the restricted direct product over the finite places.

Lemma 2.2. *There exists $\Phi_f \in \mathfrak{S}(V_{A_f})$ satisfying $\Phi_f \neq 0$, $\Phi_f|_{S_{A_f}} = 0$, and $\hat{\Phi}_f|_{S_{A_f}^*} = 0$.*

Proof. For a finite place v, take $a \in V(O_v) = O_v^n$ satisfying $|F(a)|_v = 1$ with $F = f_1 \cdots f_r$. Since $|F(a + \pi b)|_v = |F(a) + \pi c|_v = 1$ for any $b \in O_v^n$, we have $a + \pi O_v^n \subset Y_v$. Let Φ_v be the characteristic function of $a + \pi O_v^n$. Then we have $\Phi_v \in \mathfrak{S}(V_v)$, $\Phi_v \neq 0$, and $\Phi_v|_{S_v} = 0$. Take a finite place $v' \neq v$ and let $\Phi_{v'}$ be the Fourier transform of the characteristic function of $a' + \pi O_{v'}^n$ with $|F^*(a')|_{v'} = 1$. Since $\Phi_{v'} \neq 0$ and $\hat{\Phi}_{v'}|_{S_{v'}^*} = 0$, if we put $\Phi_f = \Phi_v \cdot \Phi_{v'} \cdot \prod_{w \neq v, v'} \mathrm{ch}_{O_w^n}$ where $\mathrm{ch}_{O_w^n}$ is the characteristic function of O_w^n, this satisfies our condition. □

To prove the following proposition, we shall use the argument similar to the one on p. 468 in [F. Sato 2].

Proposition 2.3. *For $\Phi \in \mathfrak{S}(V_A)$ satisfying $\Phi|_{S_A} = \hat{\Phi}|_{S_A^*} = 0$, the functions $Z_a(\omega, \Phi)$ and $Z_a^*(\hat{\omega}, \hat{\Phi})$ can be analytically continued to the whole $\Omega(k_A^\times/k^\times)^r$ and they satisfy a functional equation $Z_a(\omega, \Phi) = Z_a^*(\hat{\omega}, \hat{\Phi})$ where $\hat{\omega} = \omega_\kappa \omega^{-1}$ and $\hat{\Phi}$ is the Fourier transform of Φ.*

Proof. Note that $Z_a(\omega, \Phi)$ (resp. $Z_a^*(\hat{\omega}, \hat{\Phi})$) converges absolutely on $A = \{\omega \in \Omega(k_A^\times/k^\times)^r ; \sigma(\omega) > \kappa\}$ (resp. $A^* = \{\omega \in \Omega(k_A^\times/k^\times)^r ; \sigma(\omega) < 0\}$). Take $\omega_b \in A$ and $\omega_{b^*} \in A^*$ with $b, b^* \in \mathbb{Z}^r$. For $c = b - b^*$ and $\varepsilon = \pm 1$, put

$$Z_a^\varepsilon(\omega, \Phi) = \int_{\substack{G_A/G_k \\ \omega_c(\chi(g))^\varepsilon \geq 1}} \omega(\chi(g)) \sum_{\xi \in Y_k} \Phi(g\xi) d_{G_A}(g)$$

and

$$Z_a^{*\varepsilon}(\hat{\omega}, \hat{\Phi}) = \int_{\substack{G_A/G_k \\ \omega_c(\chi(g))^\varepsilon \geq 1}} \omega(\chi(g))\omega_\kappa(\chi^{-1}(g)) \sum_{\eta \in Y_k^*} \hat{\Phi}(g^*\eta) d_{G_A}(g).$$

We have $Z_a(\omega, \Phi) = Z_a^+(\omega, \Phi) + Z_a^-(\omega, \Phi)$ and $Z_a^*(\hat{\omega}, \hat{\Phi}) = Z_a^{*+}(\hat{\omega}, \hat{\Phi}) + Z_a^{*-}(\hat{\omega}, \hat{\Phi})$. Then $Z_a^\varepsilon(\omega, \Phi)$ converges absolutely on

$$B^\varepsilon = \{\omega \in \Omega(k_A^\times/k^\times)^r \ ; \ \omega\omega_c^{\varepsilon t} \in A \text{ for some } t \geq 0\}$$

and $Z_a^{*\varepsilon}(\hat{\omega}, \hat{\Phi})$ converges absolutely on

$$B^{*\varepsilon} = \{\omega \in \Omega(k_A^\times/k^\times)^r \ ; \ \omega\omega_c^{\varepsilon t} \in A^* \text{ for some } t \geq 0\}.$$

Note that we have $B^+ = B^{*-} = \Omega(k_A^\times/k^\times)^r$ since $\sigma(\omega_c) > 0$. For example, for any $\omega \in \Omega(k_A^\times/k^\times)^r$, take $t > 0$ satisfying $\sigma(\omega\omega_c^t) > \kappa$, i.e., $\omega\omega_c^t \in A$. Then we have

$$\infty > Z_a^+(\omega\omega_c^t, \Phi) = \int_{\substack{G_A/G_k \\ \omega_c(\chi(g)) \geq 1}} \omega_c(\chi(g))^t \cdot \omega(\chi(g)) \sum_{\xi \in Y_k} \Phi(g\xi) d_{G_A}(g)$$
$$\geq Z_a^+(\omega, \Phi).$$

On $B^\varepsilon \cap B^{*\varepsilon}$, by the adelic Poisson summation formula

$$\sum_{\xi \in Y_k} \Phi(g\xi) = \omega_\kappa(\chi^{-1}(g)) \sum_{\eta \in Y_k^*} \hat{\Phi}(g^*\eta)$$

for our Φ, we have $Z_a^\varepsilon(\omega, \Phi) = Z_a^{*\varepsilon}(\hat{\omega}, \hat{\Phi})$ so that $Z_a^\varepsilon(\omega, \Phi)$ and $Z_a^{*\varepsilon}(\hat{\omega}, \hat{\Phi})$ are analytically continued to $B^\varepsilon \cup B^{*\varepsilon} = \Omega(k_A^\times/k^\times)^r$. Hence $Z_a(\omega, \Phi) = Z_a^+(\omega, \Phi) + Z_a^-(\omega, \Phi)$ and $Z_a^*(\hat{\omega}, \hat{\Phi}) = Z_a^{*+}(\hat{\omega}, \hat{\Phi}) + Z_a^{*-}(\hat{\omega}, \hat{\Phi})$ are analytically continued to the whole $\Omega(k_A^\times/k^\times)^r$ and they satisfy the functional equation $Z_a(\omega, \Phi) = Z_a^*(\hat{\omega}, \hat{\Phi})$. □

Theorem 2.4. *For our P.V. (G, V), the Euler product $\prod_v Z_v(\omega_v, \Phi_v)$ has a functional equation:*

$$\prod_{v \in \Sigma} Z_v(\omega_v, \Phi_v) = \prod_{v \in \Sigma} Z_v^*(\hat{\omega}_v, \hat{\Phi}) \tag{2.3}$$

where $\hat{\omega}_v = (\omega_v)_\kappa \omega_v^{-1}$.

Proof. For any $\Phi_f \in \mathfrak{S}(V_{A_f})$, put $\Phi = \Phi_\infty^0 \otimes \Phi_f$ where $\Phi_\infty^0|_{S_\infty} = \hat{\Phi}_\infty^0|_{S_\infty^*} = 0$ and $\Phi_\infty^0 \neq 0$ (cf. Lemma 2.1). Then we have $\Phi|_{S_A} = \hat{\Phi}|_{S_A^*} = 0$ and by

Proposition 2.3, we obtain

$$Z_\infty(\omega_\infty, \Phi_\infty^0) \cdot Z_f(\omega_f, \Phi_f) = Z_\infty^*(\hat{\omega}_\infty, \hat{\Phi}_\infty^0) \cdot Z_f^*(\hat{\omega}_f, \hat{\Phi}_f) \quad \text{for any } \Phi_f. \tag{2.4}$$

On the other hand, for any $\Phi_\infty \in \mathfrak{S}(V_\infty)$, put $\Phi = \Phi_\infty \otimes \Phi_f^0$ with Φ_f^0 as in Lemma 2.2. Similarly we have

$$Z_\infty(\omega_\infty, \Phi_\infty) \cdot Z_f(\omega_f, \Phi_f^0) = Z_\infty^*(\hat{\omega}_\infty, \hat{\Phi}_\infty) \cdot Z_f^*(\hat{\omega}_f, \hat{\Phi}_f^0) \quad \text{for any } \Phi_\infty. \tag{2.5}$$

Multiplying equations (2.4) and (2.5), we have $Z_m(\omega, \Phi)Z_m(\omega, \Phi^0) = Z_m^*(\hat{\omega}, \hat{\Phi})Z_m^*(\hat{\omega}, \hat{\Phi}^0)$ with $\Phi = \Phi_\infty \otimes \Phi_f$ and $\Phi^0 = \Phi_\infty^0 \otimes \Phi_f^0$. By (2.4) with $\Phi_f = \Phi_f^0$, we have $Z_m(\omega, \Phi^0) = Z_m^*(\hat{\omega}, \hat{\Phi}^0)$ and hence $Z_m(\omega, \Phi) = Z_m^*(\hat{\omega}, \hat{\Phi})$ holds for any Φ. □

3. Unramified Γ-factors and Igusa local zeta functions

By Theorem 2.4, we have

$$Z_v(\omega_v, \Phi_v) \cdot \prod_{w \neq v} Z_m(\omega_w, \Phi_w) = Z_v^*(\hat{\omega}_v, \hat{\Phi}_v) \cdot \prod_{w \neq v} Z_m^*(\hat{\omega}_w, \hat{\Phi}_w). \tag{3.1}$$

$$Z_v^*(\hat{\omega}_v, \hat{\Psi}_v) \cdot \prod_{w \neq v} Z_m^*(\hat{\omega}_w, \hat{\Phi}_w) = Z_v(\omega_v, \Psi_v) \cdot \prod_{w \neq v} Z_m(\omega_w, \Phi_w). \tag{3.2}$$

Multiplying (3.1) and (3.2), we have

$$Z_v(\omega_v, \Phi_v) \cdot Z_v^*(\hat{\omega}_v, \hat{\Psi}_v) = Z_v^*(\hat{\omega}_v, \hat{\Phi}_v) \cdot Z_v(\omega_v, \Psi_v). \tag{3.3}$$

Namely, there exists $\Gamma_v(\omega_v)$ satisfying

$$Z_v(\omega_v, \Phi_v) = \Gamma_v(\omega_v) Z_v^*(\hat{\omega}_v, \hat{\Phi}_v) \quad \text{for any } \Phi_v \in \mathfrak{S}(V_v). \tag{3.4}$$

Thus we can obtain the local functional equation. However, it has already been obtained in more general cases [Igusa 1], [Gyoja].

In this section, we express a Γ-factor $\Gamma_v(\omega_v)$ by the Igusa local zeta function $Z_v(s)$ when ω_v is an unramified quasicharacter $\omega_{v,s} = (|\ |_v^{s_1}, \dots, |\ |_v^{s_r})$ for a finite place v of k. For one variable case ($r = 1$), including ramified case, see [Igusa 2].

Lemma 3.1. *For a finite place v of k, we have $Z_v(\omega_{v,s}, \mathrm{ch}_{O_v^n}) = c_v Z_v(s - \kappa)$ where $c_v^{-1} = \lim_{s_1, \dots, s_r \to \infty} Z_v(s)$.*

Proof. We write $|f(x)|_v^s = \omega_{v,s}(f(x)) = |f_1(x)|_v^{s_1} \cdots |f_r(x)|_v^{s_r}$. Then

$$Z_v(\omega_{v,s}, \mathrm{ch}_{O_v^n}) = \int_{Y_v \cap O_v^n} |f(x)|_v^s d_{Y_v}(x)$$

where d_{Y_v} is a G_v-invariant measure with $\int_{Y_v^0} d_{Y_v} = 1$. Here $Y_v^0 = \{x \in O_v^n ; |F(x)|_v = 1\}$ with $F(x) = f_1(x) \cdots f_r(x)$. Therefore if we put $d_{Y_v}(x) =$

$c_v \frac{dx}{|f(x)|_v^\kappa}$ with $\int_{O_v^n} dx = 1$, we have

$$Z_v(\omega_{v,s}, \mathrm{ch}_{O_v^n}) = c_v \int_{Y_v \cap O_v^n} |f(x)|_v^{s-\kappa} dx = c_v Z_v(s-\kappa).$$

The last equality holds when $\mathrm{Re}(s-\kappa) > 0$, i.e., $|0|_v^{s-\kappa} = 0$, and then it holds for all s by analytic continuation. We have

$$\begin{aligned} 1 &= \int_{Y_v^0} d_{Y_v} = c_v \int_{Y_v^0} dx = c_v \int_{|f(x)|_v = 1} dx \\ &= c_v \lim_{s \to \infty} \int_{O_v^n} |f(x)|_v^s dx = c_v \lim_{s\to\infty} Z_v(s) \end{aligned}$$

since $f_i(x) \in \pi O_v$ implies $\lim_{s_i \to \infty} |f_i(x)|_v^{s_i} = 0$. □

Lemma 3.2. *For a finite place v of k,*

$$Z_v^*(\hat{\omega}_{v,s}, \hat{\mathrm{ch}}_{O_v^n}) = N(d_v)^{-(d,s)+\frac{n}{2}} \cdot c_v Z_v^*(-s)$$

where d_v = the different of k_v, and $(d, s) = s_1 \deg f_1 + \cdots + s_r \deg f_r$ where $Z_v^*(s)$ is the Igusa local zeta function $Z_v(s)$ for f^*.

Proof. By using the self-dual measure, we have $\hat{\mathrm{ch}}_{O_v^n} = N(d_v)^{-\frac{n}{2}} \mathrm{ch}_{(d_v^{-1})^n}$ and hence

$$\begin{aligned} Z_v^*(\hat{\omega}_{v,s}, \hat{\mathrm{ch}}_{O_v^n}) &= Z_v^*(\omega_{v,\kappa-s}, N(d_v)^{-\frac{n}{2}} \mathrm{ch}_{(d_v^{-1})^n}) \\ &= N(d_v)^{-\frac{n}{2}} \cdot c_v \int_{Y_v^* \cap (d_v^{-1})^n} |f^*(x)|_v^{-s} dx. \end{aligned}$$

Now if $d_v^{-1} = \pi_v^{-t} O_v$, then $N(d_v) = q_v^t$. By the change of variables $x = \pi_v^{-t} y$, we have $dx = N(d_v)^n dy$ and $|f^*(x)|_v^{-s} = N(d_v)^{-(d,s)} \cdot |f^*(y)|_v^{-s}$. Hence we obtain our result. □

By these lemmas, we obtain the following theorem.

Theorem 3.3. *For any finite place v of k, we have*

$$Z_v(\omega_{v,s}, \Phi_v) = \Gamma_v(\omega_{v,s}) Z_v^*(\omega_{v,\kappa-s}, \hat{\Phi}_v)$$

for any $\Phi_v \in \mathfrak{S}(V_v)$ where Γ_v is given by $\Gamma_v(\omega_{v,s}) = N(d_v)^{(d,s)-\frac{n}{2}} \cdot Z_v(s-\kappa)/Z_v^*(-s)$ where $Z_v(s) = \int_{O_v^n} |f(x)|_v^s dx$ and $Z_v^*(s) = \int_{O_v^n} |f^*(x)|_v^s dx$ with $\int_{O_v^n} dx = 1$ is the Igusa local zeta function and $(d, s) = s_1 \deg f_1 + \cdots + s_r \deg f_r$.

4. Arithmetic calculus (example)

Theorem 4.1 (Principle of Calculus of Fourier transforms). *Let (G, ρ, V) be a reductive regular P.V. defined over an algebraic number field k satisfying $Z_a =$*

τZ_m. *Then for any* $\Phi_\infty \in \mathfrak{S}(V_\infty)$, *we have*

$$(4.1)\quad \begin{aligned} &Z_\infty(|\ \ |_\infty^s, \Phi_\infty)\cdot \prod_{v\in\Sigma_f} c_v Z_v(s-\kappa) \\ &= |D_k|^{-(d,s)+\frac{n}{2}}\cdot Z_\infty^*(|\ \ |_\infty^{\kappa-s}, \hat{\Phi}_\infty)\cdot \prod_{v\in\Sigma_f} c_v Z_v^*(-s)\end{aligned}$$

where D_k *is the discriminant of* k, $Z_v(s) =$ *the Igusa local zeta function, and* $\Sigma_f =$ *the set of finite places of* k.

Proof. By Theorem 2.4, Lemmas 3.1 and 3.2 for $\Phi = \Phi_\infty \otimes (\bigotimes_{v\in\Sigma_f} \mathrm{ch}_{O_v^n})$ and $\omega = |\ \ |_\infty^s \cdot \prod_{v\in\Sigma_f}\omega_{v,s}$, we obtain our result. □

This theorem shows that we can obtain Fourier transforms over $\mathbb{R}$ or $\mathbb{C}$ if we have the explicit form of the Igusa local zeta function. In this section, we shall calculate the Fourier transform of the complex power of the relative invariants of the following $(\tilde{G}, \rho, V)$ where $V = \{\tilde{x} = (X\,;\, y,\, z,\, w)\,;\, {}^tX = -X \in M_{2m+1},\ y,\, z,\, w \in M_{2m+1,1}\}$, and $\rho(g)\tilde{x} = (\alpha A X {}^tA\,;\, \beta A y,\, \gamma {}^tA^{-1}z,\, \delta {}^tA^{-1}w)$ for $g = (\alpha, \beta, \gamma, \delta\,;\, A) \in \tilde{G} = GL_1^4 \times SL_{2m+1}$ with $m \geq 2$. Then $\ker\rho = \{1,\, (1,\, -1,\, -1,\, -1\,;\, -I_{2m+1})\}$. Put

$$\xi = \left(\left(\begin{array}{cc|c} 0 & I_m & 0 \\ -I_m & 0 & 0 \\ \hline \multicolumn{2}{c|}{0} & 0 \end{array}\right)\,;\, e_{2m+1},\ e_1 + e_{2m+1},\ e_{m+1} + e_{2m+1}\right)$$

where

$$e_i = {}^t(0, \ldots, 0, \overset{i}{\smile}{1}, 0, \ldots, 0).$$

Then

$$f_1(\tilde{x}) = \text{Pfaffian of } \left(\begin{array}{c|c} X & y \\ \hline -{}^ty & 0\end{array}\right)$$

is a relative invariant corresponding to $\chi_1(g) = \alpha^m\beta$. Similarly, $f_2(\tilde{x}) = \langle y,\, z\rangle$ (resp. $f_3(\tilde{x}) = \langle y,\, w\rangle$, $f_4(\tilde{x}) = {}^tzXw$) is a relative invariant corresponding to $\chi_2(g) = \beta\gamma$ (resp. $\chi_3(g) = \beta\delta$, $\chi_4(g) = \alpha\gamma\delta$). Then we have $S = \{f_1 = 0\}\cup\{f_2 = 0\}\cup\{f_3 = 0\}\cup\{f_4 = 0\}$ with $V - S = \rho(G)\cdot\xi$ and a generic isotropy subgroup $\rho(G_\xi) = Sp_{m-1}$ is connected semisimple. We have $\det\rho(g)^2 = \chi_1^{2(2m-1)}\chi_2^2\chi_3^2\chi_4^{4m}$, i.e., $\kappa = (2m-1,\, 1,\, 1,\, 2m)$. By [K–K], this P.V. satisfies $Z_a = \tau Z_m$ for some $\tau > 0$.

Proposition 4.2 (Hosokawa). *Let* K *be a p-adic field with the maximal compact subring* O_K. *Let* dx *be the Haar measure on* V_K *satisfying* $\int_{V(O_K)} dx = 1$. *Then the Igusa local zeta function* $Z_K(s) = \int_{V(O_K)} |f_1(x)|_K^{s_1}\cdots|f_4(x)|_K^{s_4}dx$ *is given by*

$$(4.2)\quad \begin{aligned} Z_K(s) = &\prod_{i=1}^{m}\frac{(2i-1)}{1-q^{-(s_1+2i-1)}}\cdot\prod_{k=2}^{4}\frac{(1)}{1-q^{-(s_k+1)}} \\ &\cdot\frac{(2m)}{1-q^{-(s_4+2m)}}\cdot\frac{(2m+1)}{1-q^{-(s_5+2m+1)}}\end{aligned}$$

where $(i) = 1 - q^{-i}$, $s_5 = s_1 + \cdots + s_4$ and $q =$ the module of K.

Now we take $k = \mathbb{Q}$ in Theorem 4.1 and we have

$$\int_{V_\mathbb{R} - S_\mathbb{R}} |f(x)|_\mathbb{R}^s \hat{\Phi}_\mathbb{R}(x) dx = \gamma(s) \int_{V_\mathbb{R} - S_\mathbb{R}} |f(x)|_\mathbb{R}^{-s-\kappa} \Phi_\mathbb{R}(x)\, dx \tag{4.3}$$

with $\gamma(s) = \prod_p c_p Z_p(-s-\kappa) / \prod_p c_p Z_p(s)$ since

$$Z_\mathbb{R}(|\ |_\mathbb{R}^s, \Phi_\mathbb{R}) = \int_{V_\mathbb{R} - S_\mathbb{R}} |f(x)|_\mathbb{R}^{s-\kappa} \Phi_\mathbb{R}(x)\, dx.$$

Since

$$c_p^{-1} = \lim_{s_1, \dots, s_4 \to \infty} Z_{\mathbb{Q}_p}(s) = \prod_{i=1}^{m} (2i-1) \cdot (1)^3 \cdot (2m) \cdot (2m+1)$$

by Lemma 3.1 and Proposition 4.2, we have

$$\prod_p c_p Z_p(s) = \prod_{i=1}^{m} \zeta(s_1 + 2i - 1) \cdot \prod_{k=2}^{4} \zeta(s_k + 1) \cdot \zeta(s_4 + 2m) \cdot \zeta(s_5 + 2m + 1)$$

where $\zeta(s)$ is the Riemann zeta function. Since $-s - \kappa = (-s_1 - 2m + 1, -s_2 - 1, -s_3 - 1, -s_4 - 2m)$, we have

$$\prod_p c_p Z_p(-s-\kappa) = \prod_{i=1}^{m} \zeta(-s_1 - 2i + 2) \cdot \prod_{k=2}^{3} \zeta(-s_k) \cdot \zeta(-s_4 + 1 - 2m) \cdot \zeta(-s_4) \cdot \zeta(-s_5 - 2m).$$

Thus, by using the functional equation of the Riemann zeta function:

$$\zeta(-s)/\zeta(1+s) = (2\pi)^{-s-1} \cdot 2 \cdot \left(-\sin \frac{\pi s}{2}\right) \cdot \Gamma(1+s),$$

we have

$$\begin{aligned}
\gamma(s) &= \prod_p c_p Z_p(-s-\kappa) / \prod_p c_p Z_p(s) \\
&= \prod_{i=1}^{m} \frac{\zeta(-(s_1 + 2i - 2))}{\zeta(1 + (s_1 + 2i - 2))} \cdot \prod_{k=2}^{4} \frac{\zeta(-s_k)}{\zeta(1 + s_k)} \cdot \frac{\zeta(-(s_4 + 2m - 1))}{\zeta(1 + (s_4 + 2m - 1))} \cdot \frac{\zeta(-(s_5 + 2m))}{\zeta(1 + (s_5 + 2m))} \\
&= \prod_{i=1}^{m} (2\pi)^{-s_1 - 2i + 1} \cdot 2 \cdot \left(-\sin \frac{\pi(s_1 + 2i - 2)}{2}\right) \cdot \Gamma(s_1 + 2i - 1) \\
&\quad \cdot \prod_{k=2}^{4} (2\pi)^{-s_k - 1} \cdot 2 \cdot \left(-\sin \frac{\pi s_k}{2}\right) \cdot \Gamma(1 + s_k) \\
&\quad \cdot (2\pi)^{-s_4 - 2m} \cdot 2 \cdot \left(-\sin \frac{\pi(s_4 + 2m - 1)}{2}\right) \cdot \Gamma(s_4 + 2m) \\
&\quad \cdot (2\pi)^{-s_5 - 2m - 1} \cdot 2 \cdot \left(-\sin \frac{\pi(s_5 + 2m)}{2}\right) \cdot \Gamma(s_5 + 2m + 1)
\end{aligned}$$

with $s_5 = s_1 + s_2 + s_3 + s_4$

$$= (2\pi)^{-t} \cdot (-2)^{m+5} \cdot \prod_{i=1}^{m} \sin \frac{\pi(s_1 + 2i - 2)}{2} \cdot \prod_{k=2}^{4} \sin \frac{\pi s_k}{2} \cdot \sin \frac{\pi(s_4 + 2m - 1)}{2}$$

$$\cdot \sin \frac{\pi}{2}(s_1 + s_2 + s_3 + s_4 + 2m) \cdot \prod_{i=1}^{m} \Gamma(s_1 + 2i - 1) \cdot \prod_{k=2}^{4} \Gamma(s_k + 1)$$

$$\cdot \Gamma(s_4 + 2m) \cdot \Gamma(s_1 + s_2 + s_3 + s_4 + 2m + 1)$$

$$\text{with } t = (m+1)s_1 + 2s_2 + 2s_3 + 3s_4 + (m+2)^2.$$

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